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Skin Friction of Power Law Fluids in Turbulent Flow over a Flat Plate

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An analysis is presented for skin friction of power law fluids in turbulent flow over a flat plate. A momentum balance is combined with a logarithmic velocity profile and the resulting equation is integrated. Skin friction is shown to be a function of non-Newtonian Reynolds number and power law shear rate exponent. Closed form solutions for viscous drag are obtained for some values of shear rate exponent, but in general a numerical integration is necessary.

The equations and curves presented are valid for any power law fluid. Computations may be carried out once the fluid viscosity and shear rate exponent are known. As an example, the results are applied to a power law fluid consisting of Carbopol in water. It is shown how viscous drag is affected by changes in the polymer concentration.

Industrial processes involving non-Newtonian fluids have become increasingly important during the past decades. In the design of these processes pumping power and pressure drop may need to be determined as a function of flow rate (*1* to *6*). This need has stimulated interest in the study of laminar and turbulent flows of non-Newtonian fluids. Results of these studies (*7* to *9*) indicate that the pressure drop of non-Newtonian fluids cannot always be predicted satisfactorily from experimental correlations obtained for Newtonian fluids. In other words, the behavior of non-Newtonian fluids should be described by theories and experimental correlations developed specifically for these fluids (*8*, *9*).

Flows in pipes and along flat plates are two flow situations of practical interest. Laminar and turbulent pipe flows, and laminar flow along a flat plate (*10* to *15*) have been investigated extensively for some classes of non-Newtonian fluids. However, turbulent flow along a flat plate has received much less attention (*16*), despite the fact that in general it is of greater engineering interest than laminar flow. To fulfill an engineering need, this paper treats the turbulent flow of power law non-Newtonian fluids along a flat plate.

Velocity data for non-Newtonian fluids in turbulent pipe flow were obtained by several investigators (*9, 17* to *19*). They represented the rheological behavior of these fluids by a power law model and correlated the velocity profiles by a logarithmic function (*9, 17, 20*). In this paper such a logarithmic profile is used with a boundary-layer momentum balance to obtain drag on a flat plate. The profile used is written in a form that can be adjusted to fit experimental data for any specific power law fluid. The resulting equation for drag on a flat plate can therefore be applied to any fluid whose rheological properties can be described by the power law model.

In the following sections the boundary-layer momentum equation is combined with the logarithmic profile and is integrated. The resulting equation for flat plate drag is then applied to a specific fluid to demonstrate the use of the derived drag prediction.

VELOCITY PROFILES FOR POWER LAW FLUIDS

Many non-Newtonian fluids can be characterized satisfactorily by a power law expression over large ranges of shear rate (*9, 17, 20, 21*). This expression is given by

$$\tau = k\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{1}$$

When the power law exponent n is unity, Equation (1) reduces to Newton's law of viscosity.

Pressure drop for Newtonian pipe flow is correlated by the equation

$$\Delta p = 4f\frac{L}{D}\frac{\rho V^2}{2}$$

where the friction factor is defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho V^2}$$

For laminar flow the friction factor is

$$f = \frac{16}{N_{Re}} = \frac{16\,k}{\rho V D} \tag{2}$$

It can be shown (*8*) that the friction factor in the case of laminar flow of a power law fluid is

$$f = \frac{16\,k}{\rho V^{2-n} D^n}\,\frac{\left(\dfrac{3n+1}{4n}\right)^n}{8^{1-n}} \tag{3}$$

From a comparison of Equations (2) and (3) it has become common practice to define a Reynolds number for a power law fluid as

$$N_{Re_D} = \frac{\rho V^{2-n} D^n}{k}\,\frac{8^{1-n}}{\left(\dfrac{3n+1}{4n}\right)^n} \tag{4}$$

This definition of Reynolds number is used both for laminar and turbulent flows.

For turbulent pipe flow of Newtonian fluids the velocity profile is correlated by the well-known logarithmic profile:

$$\frac{u}{u^*} = A\ln\frac{yu^*\rho}{k} + B \tag{5}$$

After studying pipe flow of several power law fluids, Bogue and Metzner (*17*) showed that with the power law model and the definition

$$\eta = \frac{y^n (u^*)^{2-n}\rho}{k} \tag{6}$$

velocity profiles of power law fluids in turbulent flow could be correlated by a function of the form

$$\phi = \frac{u}{u^*} = A\ln\,\eta + B \tag{7}$$

which is seen to be similar to Equation (5).

In 1921 Prandtl (*22*) showed that the velocity profile given in Equation (5) for pipe flow could be used successfully to predict boundary-layer growth along a flat plate. By following Prandtl's approach, the power law profile of Equation (7) is used in this paper to integrate the momentum equation.

TURBULENT BOUNDARY-LAYER DEVELOPMENT

To determine boundary-layer growth for turbulent flow along a flat plate, consider a momentum balance relating friction drag to momentum flux. For the control volume shown in Figure 1, Schlichting (*23*) shows that the shear stress at the wall of a flat plate held parallel to a flow with uniform free stream velocity U is equal to

$$\tau_w(x) = \rho\frac{d}{dx}\int_0^\infty u(U-u)\,dy$$

Because the integrand is assumed to be zero for values of y greater than the boundary-layer thickness δ, this equation may be written as

$$\tau_w(x) = \rho U^2\frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy \tag{8}$$

To integrate Equation (8) the velocity profile of Equation (7) is substituted into Equation (8). It is noted from Equations (6) and (7) that

$$dy = \frac{1}{n}\left(\frac{k}{\rho u^{*2-n}}\right)^{\frac{1}{n}}\eta^{\frac{1-n}{n}}\,d\eta$$

$$= \frac{1}{n}\left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}}\Phi^{\frac{2-n}{n}}\,\eta^{\frac{1-n}{n}}\,d\eta$$

The physically important limits of integration are $u = 0$ at the wall and $u = U$ at $y = \delta$. However, the velocity profile of Equation (7) gives a negative velocity at the wall; the integration is therefore carried out from the point where $u = 0$, where by definition $\eta = \eta_w$, to the edge of the boundary layer, with the result that

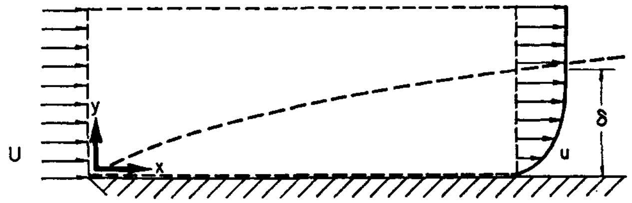

Fig. 1. Control volume and coordinate system for momentum analysis.

$$\tau_w(x) = \frac{\rho U^2}{n} \frac{d}{dx} \int_{\eta_w}^{\eta_1} \frac{\phi}{\Phi}\left(1 - \frac{\phi}{\Phi}\right) \left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}} \Phi^{\frac{2-n}{n}} \eta^{\frac{1-n}{n}} d\eta$$

The fluid properties n, k and ρ and free stream velocity U are assumed to be independent of position along the plate, so that

$$\tau_w(x) = \frac{\rho U^2}{n} \left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}} \frac{d}{dx} \int_{\eta_w}^{\eta_1} \left(\phi - \frac{\phi^2}{\Phi}\right) \Phi^{\frac{2(1-n)}{n}} \eta^{\frac{1-n}{n}} d\eta$$

Carrying out the differentiation by application of Leibnitz' rule for differentiating a definite integral, one obtains

$$\tau_w(x) = \frac{\rho U^2}{n^2} \left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}} \frac{d\eta_1}{dx} \frac{d\Phi}{d\eta_1} \left\{2(1-n)\, \Phi^{\frac{2-3n}{n}} \int_{\eta_w}^{\eta_1} \phi\, \eta^{\frac{1-n}{n}} d\eta + (3n-2)\, \Phi^{\frac{2-4n}{n}} \int_{\eta_w}^{\eta_1} \phi^2\, \eta^{\frac{1-n}{n}} d\eta \right\} \quad (9)$$

In the differentiation it is important to note that η_1 and Φ depend on x, but that ϕ is a function of y only.

For convenience in integration, the variables in Equation (9) are transformed by introducing

$$\phi = Bz, \text{ where } z = 1 + \frac{A}{B} \ln \eta \quad (10)$$

After integrating once, and collecting terms, there is obtained

$$\tau_w(x) = A\rho U^2 \left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}} \frac{d\eta_1}{dx} \frac{d\Phi}{d\eta_1} \left\{ 2(1-n)\, \Phi^{\frac{2-3n}{n}} \left\{\eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right) - 1\right] + e^{-\frac{B}{nA}}\right\} + (3n-2)(nA)\Phi^{\frac{2-4n}{n}} \left\{ \eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right)^2 - 2\left(\frac{\Phi}{nA}\right) + 2\right] - 2e^{-\frac{B}{nA}} \right\}\right\} \quad (11)$$

The constant terms in front of the first brace in Equation (11) may be rearranged into the non-Newtonian grouping of Equation (6) by utilizing first the definition of u^* in the form

$$\tau_w(x) = \rho u^{*2} = \frac{\rho U^2}{\Phi^2}$$

Substitution of this result into Equation (11) gives

$$\left[\frac{\rho U^{2-n}}{k}\right]^{\frac{1}{n}} = A \frac{d\eta_1}{dx} \frac{d\Phi}{d\eta_1} \left\{ 2(1-n)\, \Phi^{\frac{2-n}{n}} \left\{\eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right) - 1\right] + e^{-\frac{B}{nA}}\right\} + (3n-2)(nA)\Phi^{\frac{2-2n}{n}} \left\{ \eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right)^2 - 2\left(\frac{\Phi}{nA}\right) + 2\right] - 2e^{-\frac{B}{nA}} \right\}\right\} \quad (12)$$

It is noted that

$$\int_0^x \left[\frac{\rho U^{2-n}}{k}\right]^{\frac{1}{n}} dx = \left[\frac{\rho x^n U^{2-n}}{k}\right]^{\frac{1}{n}} = \left(\frac{3n+1}{4n}\right) 8^{\frac{n-1}{n}} [N_{Re_x}]^{\frac{1}{n}} \quad (13)$$

By using this result, Equation (12) may be rearranged into the form

$$[N_{Re_x}]^{\frac{1}{n}} = \frac{A}{\left(\frac{3n+1}{4n}\right) 8^{\frac{n-1}{n}}} \int_{\eta_w}^{\eta_1} \frac{d\Phi}{d\eta_1} \left\{ 2(1-n)\Phi^{\frac{2-n}{n}} \left\{\eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right) - 1\right] + e^{-\frac{B}{nA}}\right\} + (3n-2)(nA)\Phi^{\frac{2-2n}{n}} \left\{ \eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right)^2 - 2\left(\frac{\Phi}{nA}\right) + 2\right] - 2e^{-\frac{B}{nA}} \right\}\right\} d\eta_1$$

After considerable algebra, the result of this integration may be written as

$$N_{Re_x} = \frac{(nA)^{2+n} e^{-\frac{B}{A}}}{\left(\frac{3n+1}{4n}\right)^n 8^{n-1}} \left(\frac{\Phi}{nA}\right)^{2-n} \left\{ (1-n)\frac{\Phi}{nA} + \frac{4-6n}{2-n} + e^{\frac{\Phi}{nA}} \left[\frac{\Phi}{nA} - 4\right] + \frac{2n+4}{n}\left(\frac{B}{\Phi}\right)^{\frac{2-n}{n}} \int_0^{\frac{\Phi}{B}} z^{\frac{2-2n}{n}} e^{\frac{Bz}{nA}} dz \right\}^n \quad (14)$$

Equation (14) expresses the Reynolds number N_{Re_x} in terms of the local boundary-layer thickness parameter Φ. The integral in Equation (14) can be obtained in closed form for only certain values of n (n = 1, 0.66, and 0.5), but it may easily be computed numerically for other values of n.

The viscous drag on a plate of length L is given by

$$\text{Drag} = \int_0^L \tau_w(x)\, w\, dx$$

and therefore from Equation (11)

$$\text{Drag} = A\rho U^2 \left[\frac{k}{\rho U^{2-n}}\right]^{\frac{1}{n}} \int_{\eta_w}^{\eta_1} \frac{d\Phi}{d\eta_1} \left[2(1-n)\,\Phi^{\frac{2-3n}{n}} \left\{ \eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right) - 1 \right] + e^{-\frac{B}{nA}} \right\} \right.$$

$$+ (3n-2)(nA)\Phi^{\frac{2-4n}{n}} \left\{ \eta_1^{\frac{1}{n}} \left[\left(\frac{\Phi}{nA}\right)^2 - 2\left(\frac{\Phi}{nA}\right) + 2 \right] - 2e^{-\frac{B}{nA}} \right\} \Bigg]\, w\, d\eta_1 \quad (15)$$

The drag may be expressed in nondimensional form by means of a skin friction coefficient, defined as

$$c_f = \frac{\text{Drag}}{\frac{1}{2}\rho U^2 wx} \quad (16)$$

Upon integrating Equation (15) for the drag, the coefficient of skin friction reduces to

$$c_f = \frac{2\left(\frac{\Phi}{nA}\right)\left[1 + e^{\frac{\Phi}{nA}}\right] + 4\left[1 - e^{\frac{\Phi}{nA}}\right]}{(nA)^2\left(\frac{\Phi}{nA}\right)^2 \left\{ (1-n)\frac{\Phi}{nA} + \frac{4-6n}{2-n} + e^{\frac{\Phi}{nA}}\left[\left(\frac{\Phi}{nA}\right) - 4\right] + \frac{2n+4}{n}\left(\frac{B}{\Phi}\right)^{\frac{2-n}{n}} \int_0^{\frac{\Phi}{B}} z^{\frac{2-2n}{n}} e^{\frac{Bz}{nA}} dz \right\}} \quad (17)$$

Equations (14) and (17) are the basic equations for the Reynolds number and friction coefficient of all power law fluids in turbulent flow over a flat plate.

Equations (14) and (17) still contain the integral

$$\int_0^{\frac{\Phi}{B}} z^{\frac{2-2n}{n}} e^{\frac{Bz}{nA}} dz$$

As mentioned above, this integral may be evaluated in closed form for values of n equal to 0.5, 0.66, and 1.0. For example, when $n = 0.5$, the integral is

$$\int_0^{\frac{\Phi}{B}} z^2 e^{\frac{2B}{A}z} dz = e^{\frac{2\Phi}{A}} \left[\left(\frac{\Phi}{B}\right)^2 \frac{A}{2B} - \frac{A^2}{2B^2}\left(\frac{\Phi}{B}\right) + \frac{A^3}{4B^3} \right] - \frac{A^3}{4B^3}$$

When this value is substituted into Equations (14) and (17), they become

$$N_{Re_x} = \frac{2A^{\frac{5}{2}} e^{-\frac{B}{A}}}{5^{\frac{1}{2}}} \left(\frac{\Phi}{A}\right)^{\frac{3}{2}} \left\{ 2\left(\frac{\Phi}{A}\right) + \frac{4}{3} - \frac{5}{\left(\frac{\Phi}{A}\right)^3} + e^{\frac{2\Phi}{A}} \left[4\left(\frac{\Phi}{A}\right) - 8 + \frac{10}{\left(\frac{\Phi}{A}\right)} - \frac{10}{\left(\frac{\Phi}{A}\right)^2} + \frac{5}{\left(\frac{\Phi}{A}\right)^3} \right] \right\}^{\frac{1}{2}} \quad (18)$$

$$c_f = \frac{8\left\{ \left(\frac{\Phi}{A}\right)\left[1 + e^{\frac{2\Phi}{A}}\right] + 1 - e^{\frac{2\Phi}{A}} \right\}}{A^2\left(\frac{\Phi}{A}\right)^2 \left\{ 2\left(\frac{\Phi}{A}\right) + \frac{4}{3} - \frac{5}{\left(\frac{\Phi}{A}\right)^3} + e^{\frac{2\Phi}{A}} \left[4\left(\frac{\Phi}{A}\right) - 8 + \frac{10}{\left(\frac{\Phi}{A}\right)} - \frac{10}{\left(\frac{\Phi}{A}\right)^2} + \frac{5}{\left(\frac{\Phi}{A}\right)^3} \right] \right\}} \quad (19)$$

These equations are still in general form and contain the velocity profile parameters A and B from Equation (7). In order to evaluate Reynolds number and skin friction coefficient, values for the parameters A and B must be chosen.

POWER LAW FLUID CORRELATION

For Newtonian fluids the universal velocity profile for pipe flow has been found experimentally (*24*) to be

$$\phi = 2.5 \ln\eta + 5.5$$

Bogue and Metzner (*17*) found the same form for the equation of the universal velocity profile and correlated their data for a non-Newtonian fluid consisting of water with a small amount of the additive Carbopol (a carboxypolymethylene compound). Their first correlation was

$$\phi = A \ln\eta + B$$

where $A = 2.42$ and $B = 5.57$.

They improved the correlation by making the parameter B a function of Reynolds number and power law exponent. For the numerical calculations of this paper, velocity profile data for variable B have been fitted with the expression

$$\phi = 2.42 \ln\eta + 5.57 \left[1.005 + 0.953\,(\log_{10} n) \log_{10}\left(\frac{N_{Re_x}}{10^5}\right) \right] \quad (20)$$

By using the velocity profile of Equation (20), the integrals in Equations (14) and (17) may be evaluated numerically. The results of this calculation, carried out for several values of n, are plotted in Figure 2.

In Figure 2 the total skin friction coefficient of a flat plate is presented as a function of the non-Newtonian Reynolds number with the power law exponent n as a parameter. The overall trend of the curves is toward

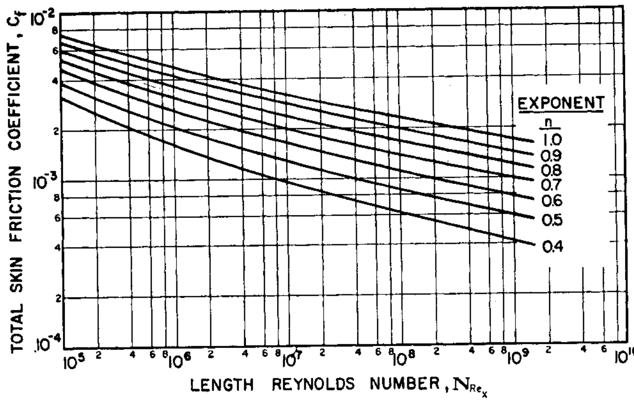

Fig. 2. Total skin friction as a function of N_{Re_x} and n for the velocity profile of Equation (20).

lower skin friction for increasing Reynolds numbers. In this respect the non-Newtonian results follow the trend of skin friction for Newtonian fluids.

When n equals unity, results of the present analysis reduce to the case of Newtonian flow. In this case, calculated skin friction coefficients are within 10% of experimentally observed values (25) over a Reynolds number range from 10^6 to 10^8. Thus the theory presented in this paper correctly describes the behavior of the limiting case of Newtonian fluids.

To characterize a power law fluid, both the power law exponent n and the viscosity k must be specified. Consequently the Reynolds number and therefore the skin friction coefficient depend on both these parameters. For example, Bogue and Metzner (*17*) show that addition of 0.3 wt. % of Carbopol to water causes a decrease in the exponent from 1.0 to 0.745 and an increase in viscosity from 2.1×10^{-5} slug/(ft.) (sec.$^{2-n}$) to 1.1×10^{-3} slug/(ft.) (sec.$^{2-n}$). In this case, reduction in exponent is accompanied by a fifty-fold increase in viscosity. Although a decrease in exponent would decrease skin friction, the increase in viscosity overshadows this effect. As a result, for an initial Newtonian Reynolds number of 10^8, the addition of 0.3% of Carbopol to water causes an increase in the skin friction coefficient.

To compute the change in skin friction for a specific change in the power law exponent and viscosity, Equations (14) and (17) should be used. It is evident that to determine the magnitude of the change, the integrals in Equations (14) and (17) must be evaluated numerically for most values of n.

CONCLUDING REMARKS

A relationship has been presented for skin friction drag of any power law fluid in turbulent flow over a flat plate. Friction has been shown to be a function of Reynolds number, viscosity k, and power law exponent n. Because a decrease in the exponent is usually accompanied by an increase in viscosity, only certain combinations of exponent and viscosity give a drag reduction for a particular flow configuration.

Curves of skin friction drag are presented for typical values of fluid parameters and equations are presented that permit numerical calculation of skin friction drag for any power law fluid flowing over a flat plate.

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NOTATION

A = velocity profile constant, defined in Equation (5)
B = velocity profile constant, defined in Equation (5)
c_f = skin friction coefficient, defined in Equation (16)
D = pipe inside diameter, ft.
f = friction factor, $\tau_w/\frac{1}{2}\rho V^2$, dimensionless
k = absolute viscosity coefficient, slugs/(ft.)(sec.$^{2-n}$)
L = length of pipe or plate, ft.
N_{Re} = Reynolds number for Newtonian fluids, $\rho VD/k$
N_{ReD}, N_{Re_x} = Reynolds numbers for non-Newtonian fluids, defined in Equations (4) and (13), respectively
n = power law exponent, dimensionless
p = static pressure, lb./sq.ft.
u = fluid velocity in boundary layer, ft./sec.
u^* = friction velocity, $\sqrt{\tau_w/\rho}$, ft./sec.
U = free stream velocity, ft./sec.
V = average velocity in pipe, ft./sec.
w = width of plate, ft.
x = coordinate parallel to plate, ft.
y = coordinate perpendicular to plate, ft.
z = transformed velocity function, defined in Equation (10)

Greek Letters

δ = boundary-layer thickness, ft.
η = nondimensional distance, $\rho u^* y/k$
ρ = density, slugs/cu.ft.
τ = shear stress, lb./sq.ft.
ϕ = nondimensional velocity, u/u^*
Φ = nondimensional velocity, U/u^*

Subscripts

1 = edge of boundary layer
w = wall

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